MINISTÉRIO DA CIÊNCIA E TECNOLOGIA
INSTITUTO NACIONAL DE PESQUISAS ESPACIAIS

INPE-11285-PRE/6722

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INPE
São José dos Campos
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Marcelo Lopes de Oliveira e Souza
Instituto Nacional de Pesquisas Espaciais – INPE
C.P. 515, 12245-970, São José dos Campos, SP, Brazil
evandro@dem.inpe.br; prado@dem.inpe.br; marcelo@dem.inpe.br

ABSTRACT

In this work, we consider the problem of two-impulse orbital transfers between coplanar elliptical orbits with minimum fuel consumption but with a time limit for this transfer. This time limit imposes a new characteristic to the problem that rules out the majority of transfer methods available in the literature. Then, as a first method (Rocco[1], Rocco et al.[2]), we used the equations presented by Lawden[3]. Those equations furnishes the optimal transfer orbit with fixed time for this transfer, between two elliptical coplanar orbits considering fixed terminal points. We adapted the method to cases with free terminal points and solved those equations to develop a software for orbital maneuvers. As a second method (Rocco[1], Rocco et al.[4]), we used the equations presented by Eckel and Vinh[5], that provides the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time transfer; or minimum time transfer for a prescribed fuel consumption; considering free terminal points. But in this work we consider only the problem with fixed time transfer, the case of minimum time for a prescribed fuel consumption was already studied in Rocco et al.[6] Then, we modified the method to consider cases of coplanar orbital transfer, and develop a software for orbital maneuvers. Therefore, we have two software that solve the same problem using different methods. The first method uses the primer vector theory. The second method uses the ordinary theory of maxima and minima. So, to test the methods we choose the same terminal orbits and the same time as input. We could verify that we didn't obtain exactly the same result. Therefore, comparing and testing the results we could select the best method to solve the problem of coplanar orbital transfer with time limit.

1 - INTRODUCTION

The majority of the spacecrafts that have been placed in orbit around the Earth utilize the basic concept of orbital transfers. During the launch, the spacecraft is placed in a parking orbit distinct from the final orbit, which the spacecraft was designed. Therefore, to reach the desired final orbit the spacecraft must perform orbital transfer maneuvers. Beyond this, the spacecraft orbit must be corrected periodically because there are perturbations acting on the spacecraft. In Brazil, we are going to have important applications with the launch of Remote Sensing Satellites RSS1 and RSS2 that belongs to the Complete Brazilian Space Mission and with the launch of the China Brazil Earth Resources Satellites CBERS1 and CBERS2.

In this work, we consider the problem of two-impulse orbital transfers between coplanar elliptical orbits with minimum fuel consumption but with a time limit for this transfer. This time limit imposes a new characteristic to the problem that rules out the majority of transfer methods. Then, we used the equations presented by Lawden[3] (method 1) and by Eckel and Vinh[5] (method 2), modified them, considering cases with different geometries, and solved those equations to develop two software for orbital maneuvers. The original methods developed by Lawden and by Eckel and Vinh were presented without numerical results. Thus, the modifications considering cases with more complex geometry, the implementation and the solutions using these methods are contributions of this work.

2 - PRESENTATION OF THE METHODS

2.1- Method 1

The bases for this method are the equations presented by Lawden[3], that furnish: the transfer orbit between coplanar elliptical orbits with minimum fuel and fixed time transfer. The equations were presented in the literature but the method was not implemented neither tested by Lawden[3] in that paper. Thus, the extension to other cases, the implementation and the solutions using this method are contributions of this work. Therefore, the method was implemented to develop a software for orbital maneuvers. By varying the time spent in the maneuver the software developed furnishes a set of results that are the solution of the problem of bi-impulsive optimal orbital transfer with time limit.

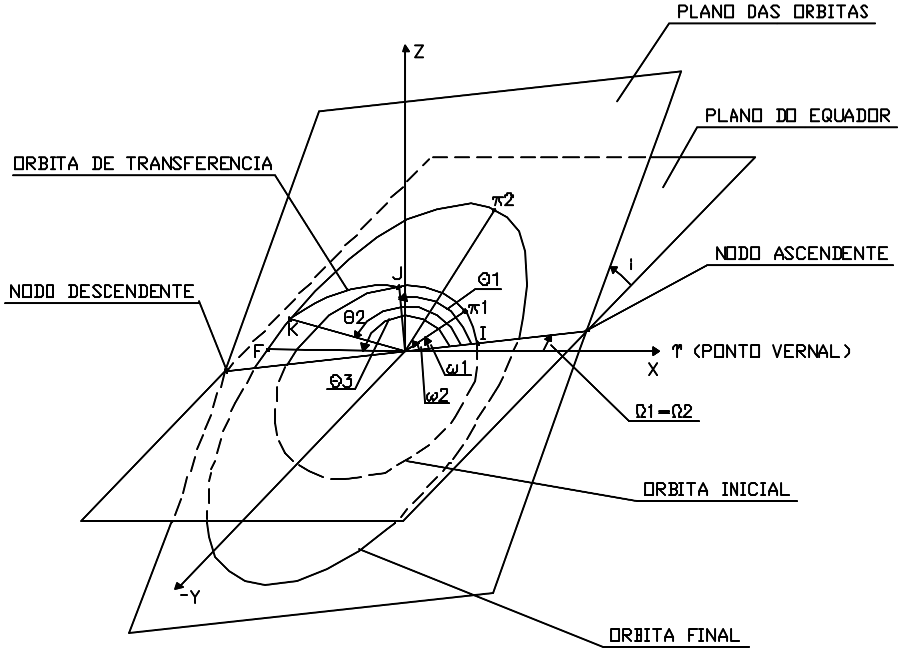


Fig. 1 – Geometry of the Maneuver

Given two coplanar terminal orbits, as showed in figure 1, we desire to obtain a transfer orbit with minimum fuel consumption and fixed time transfer. The orbits are specified by their orbital elements (subscript 1: initial orbit; subscript 2: final orbit; no subscript: transfer orbit):

a	= Semi-major axis;	ω	= Longitude of the periapsis;
e	= Eccentricity;	i	= Inclination;
l	= Semi-latus rectum;	Ω	= Longitude of the ascending node;

From Lawden[3] we have an equation system composed by twelve equations which represent the optimal conditions:

$$\Delta t_1 + \Delta t + \Delta t_2 = T \text{ (in case 1)} \tag{1a}$$

$$\Delta t_1 + \Delta t = T \text{ (in case 2)} \tag{1b}$$

$$\Delta t + \Delta t_2 = T \text{ (in case 3)} \tag{1c}$$

$$\Delta t = T \text{ (in case 4)} \tag{1d}$$

$$e_1 \cos(\theta_1 - \omega_1) = l_1 s_1 - 1 \tag{2}$$

$$e \cos(\theta_1 - \omega) = l s_1 - 1 \tag{3}$$

$$e_1 sin(\theta_1 - \omega_1) = \left(l_1 s_1 - l_1^{1/2} z_1\right) \tan \phi_1 \tag{4}$$

$$e sin(\theta_1 - \omega) = \left(l s_1 - l^{1/2} z_1\right) \tan \phi_1 \tag{5}$$

$$\left(z_1 - \frac{s_1}{z_1}\right) sin\, \phi_1 - \frac{e\alpha}{l^2 s_1 z_1} = \left(z_2 - \frac{s_2}{z_2}\right) sin\, \phi_2 - \frac{e\alpha}{l^2 s_2 z_2} \tag{6}$$

$$M_1 = M_2 - \frac{3\mu^{1/2} \alpha e^2 t}{l^{3/2}\left(1 - e^2\right)} \tag{7}$$

$$N_1 = N_2 + \frac{3\mu^{1/2} \alpha e t}{l^{3/2}\left(1 - e^2\right)} \tag{8}$$

$$e_2 \cos(\theta_2 - \omega_2) = l_2 s_2 - 1 \tag{9}$$

$$e \cos(\theta_2 - \omega) = l s_2 - 1 \tag{10}$$

$$e_2 sin(\theta_2 - \omega_2) = \left(l_2 s_2 - l_2^{1/2} z_2\right) \tan \phi_2 \tag{11}$$

$$e sin(\theta_2 - \omega) = \left(l s_2 - l^{1/2} z_2\right) \tan \phi_2 \tag{12}$$

Where:

$$M_1 = \cos(q_1 - w - f_1) + \frac{1}{l^{1/2} z_1}\cos(q_1 - w)\cos f_1$$
$$+ a\left\{\frac{\cot(q_1 - w)}{l^{3/2} s_1 z_1} + \frac{1}{1-e^2}\left[\frac{2e}{l s_1 sin(q_1 - w)} - \cot(q_1 - w)\right]\right\} \tag{13}$$

$$M_2 = \cos(q_2 - w - f_2) + \frac{1}{l^{1/2} z_2}\cos(q_2 - w)\cos f_2$$
$$+ a\left\{\frac{\cot(q_2 - w)}{l^{3/2} s_2 z_2} + \frac{1}{1-e^2}\left[\frac{2.e}{l s_2 sin(q_2 - w)} - \cot(q_2 - w)\right]\right\} \tag{14}$$

$$N_1 = \cos f_1\left(1 + \frac{l s_1 + 1}{l^{1/2} z_1}\right) + a\left\{\frac{l s_1 + 1}{l^{3/2} s_1 z_1 sin(q_1 - w)} + \frac{1}{1-e^2}\left[e\cot(q_1 - w) - \frac{2}{l s_1 sin(q_1 - w)}\right]\right\} \tag{15}$$

$$N_2 = \cos f_2\left(1 + \frac{l s_2 + 1}{l^{1/2} z_2}\right) + a\left\{\frac{l s_2 + 1}{l^{3/2} s_2 z_2 sin(q_2 - w)} + \frac{1}{1-e^2}\left[e\cot(q_2 - w) - \frac{2}{l s_2 sin(q_2 - w)}\right]\right\} \tag{16}$$

Solving this equation system by the Newton Raphson Method (cf. Rocco[1]), we obtain the transfer orbit that performs the maneuver spending a specific time but with minimum fuel consumption.

2.2- Method 2

The bases for this method are the equations presented by Eckel and Vinh[5], that furnish: the transfer orbit between non-coplanar elliptical orbits with minimum fuel and fixed time transfer; or minimum time transfer for a prescribed fuel consumption; but in this work we consider only the problem with fixed time transfer. The equations were presented in the literature but the method was not implemented neither tested by Eckel and Vinh[5] in that paper. Thus, the modifications considering the coplanar case and other cases with more complex geometries, the implementation and the solutions using this method are contributions of this work. Therefore, the method was implemented to develop a software for orbital maneuvers. By varying the time spent in the maneuver the software developed furnishes a set of results that are the solution of the problem of bi-impulsive optimal orbital transfer with time limit.

Given two terminal orbits we desire to obtain a transfer orbit with minimum fuel consumption and fixed time transfer. The orbits are specified by their orbital elements (subscript 1: initial orbit; subscript 2: final orbit; no subscript: transfer orbit):

- a = Semi-major axis;
- e = Eccentricity;
- p = Semi-latus rectum;
- w = Longitude of the periapsis;
- i = Inclination;
- W = Longitude of the ascending node;
- M = Mean anomaly;
- E = Eccentric anomaly.

The geometry of the maneuver is shown in Figure 2.

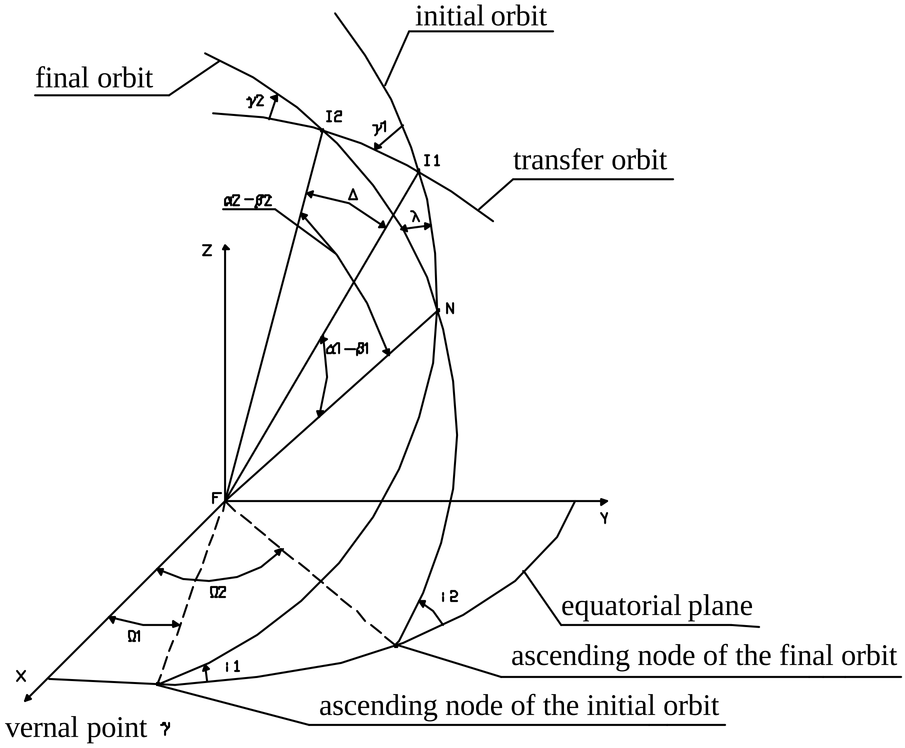


Fig. 2 – Geometry of the Maneuver.

From Eckel and Vinh[5] we have an equation system composed by three equations which represent the optimal conditions:

$$T - T_0 = 0 \tag{17}$$

$$\begin{aligned}&\left(X_1 + YZe\sin f_2\right)\left(S_1 q_1 - T_1 e\sin f_1\right) + S_1 T_1 \\ &+ W_1\left(\frac{W_1 - W_2}{\sin D} q_2 - W_1 \tan\frac{D}{2}\right) - \frac{W_1 Z e r_1 e_1 \sin \boldsymbol{a}_1}{q_1 p_1 \sin f_1 \sin \boldsymbol{g}_1} = 0\end{aligned} \tag{18}$$

$$\begin{aligned}&\left(X_2 + YZe\sin f_1\right)\left(S_2 q_2 - T_2 e\sin f_2\right) + S_2 T_2 \\ &+ W_2\left(\frac{W_2 - W_1}{\sin D} q_1 - W_2 \tan\frac{D}{2}\right) + \frac{W_2 Z e r_2 e_2 \sin a_2}{q_2 p_2 \sin f_2 \sin \boldsymbol{g}_2} = 0\end{aligned} \tag{19}$$

For the coplanar case we should substitute the two previous equations by the following:

$$\left(X_1 + YZe\sin f_2\right)\left(S_1 q_1 - T_1 e\sin f_1\right) + S_1 T_1 - \sqrt{\frac{m}{p_1}}\frac{Ze\, e_1 \sin \boldsymbol{a}_1}{q_1 V_1 \sin f_1} = 0 \tag{20}$$

$$\left(X_2 + YZe\sin f_1\right)\left(S_2 q_2 - T_2 e\sin f_2\right) + S_2 T_2 - \sqrt{\frac{m}{p_2}}\frac{Ze\, e_2 \sin \boldsymbol{a}_2}{q_2 V_2 \sin f_2} = 0 \tag{21}$$

Where:

$$b_1 = \arctan\left[\frac{\sin(W_2 - W_1)\tan(180^\circ - i_2)}{\sin i_1 + \tan(180^\circ - i_2)\cos i_1 \cos(W_2 - W_1)}\right] - w_1 \tag{22}$$

$$b_2 = \arctan\left[\frac{sin(W_2 - W_1)\tan i_1}{sini_2 + \tan i_1 \cos(180^\circ - i_2)\cos(W_2 - W_1)}\right] - w_2 \tag{23}$$

$$l = \arcsin\left[\frac{sin(W_2 - W_1)sini_1}{sin(w_2 + b2)}\right] = \arcsin\left[\frac{sin(W_2 - W_1)sini_2}{sin(w_1 + b_1)}\right] \tag{24}$$

$$\cos D = \cos(b_1 - a_1)\cos(a_2 - b_2) + sin(b_1 - a_1)sin(a_2 - b_2)\cos(180^\circ - l) \tag{25}$$

$$sinD = \frac{sin(a_2 - b_2)sin(180^\circ - l)}{sinB} \tag{26}$$

$$B = \arctan\left[\frac{sin(180^\circ - l)}{sin(b_1 - a_1)\cot(a_2 - b_2) - \cos(b_1 - a_1)\cos(180^\circ - l)}\right] \tag{27}$$

$$r_1 = \frac{p_1}{1 + e_1 \cos a_1}\,; \quad r_2 = \frac{p_2}{1 + e_2 \cos a_2} \tag{28}$$

$$f_1 = \mathrm{arc\,tg}\left[\cot D - \frac{r_1(p - r_2)}{r_2(p - r_1)\,\mathrm{sen}\,D}\right]; \quad f_2 = \mathrm{arc\,tg}\left[\frac{r_2(p - r_1)}{r_1(p - r_2)\,\mathrm{sen}\,D} - \mathrm{cotg}\,D\right] \tag{29}$$

$$p = \frac{r_1 r_2 (\cos f_1 - \cos f_2)}{r_1 \cos f_1 - r_2 \cos f_2} \tag{30}$$

$$e = \frac{r_2 - r_1}{r_1 \cos f_1 - r_2 \cos f_2} \tag{31}$$

$$a = \frac{p}{1 - e^2} \tag{32}$$

$$g_1 = \mathrm{arc\,sen}\left[-\frac{\mathrm{sen}(b_2 - a_2)}{\mathrm{sen}\,D}\,\mathrm{sen}\,f\right]; \quad g_2 = \mathrm{arc\,sen}\left[-\frac{\mathrm{sen}(b_1 - a_1)}{\mathrm{sen}\,D}\,\mathrm{sen}\,f\right] \tag{33}$$

$$x_1 = \sqrt{m}\left(\frac{e}{\sqrt{p}}\,\mathrm{sen}\,f_1 - \frac{e_1}{\sqrt{p_1}}\,\mathrm{sen}\,a_1\right); \quad x_2 = \sqrt{m}\left(\frac{e_2}{\sqrt{p_2}}\,\mathrm{sen}\,a_2 - \frac{e}{\sqrt{p}}\,\mathrm{sen}\,f_2\right) \tag{34}$$

$$y_1 = \frac{\sqrt{m}}{r_1}\left(\sqrt{p} - \sqrt{p_1}\cos g_1\right); \quad y_2 = \frac{\sqrt{m}}{r_2}\left(\sqrt{p_2}\cos g_2 - \sqrt{p}\right) \tag{35}$$

$$z_1 = \frac{\sqrt{m.p_1}}{r_1}\,\mathrm{sen}\,\boldsymbol{g}_1; \quad z_2 = \frac{\sqrt{m p_2}}{r_2}\,\mathrm{sen}\,\boldsymbol{g}_2 \tag{36}$$

$$h_1 = \left(y_1{}^2 + z_1{}^2\right)^{1/2}; \quad h_2 = \left(y_2{}^2 + z_2{}^2\right)^{1/2} \tag{37}$$

$$V_1 = \left(x_1{}^2 + h_1{}^2\right)^{1/2}; \quad V_2 = \left(x_2{}^2 + h_2{}^2\right)^{1/2} \tag{38}$$

$$S_1 = \frac{x_1}{V_1}; \quad S_2 = \frac{x_2}{V_2} \tag{39}$$

$$T_1 = \frac{y_1}{V_1}; \quad T_2 = \frac{y_2}{V_2} \tag{40}$$

$$W_1 = \frac{z_1}{V_1}; \quad W_2 = \frac{z_2}{V_2} \tag{41}$$

$$q_1 = \frac{p}{r_1}; \quad q_2 = \frac{p}{r_2} \tag{42}$$

$$E_1 = \mathrm{arc}\cos\left(\frac{e + \cos f_1}{1 + e\cos f_1}\right); \quad E_2 = \mathrm{arc}\cos\left(\frac{e + \cos f_2}{1 + e\cos f_2}\right) \tag{43}$$

$$\mathrm{sen}\,E_1 = \frac{\sqrt{1-e^2}\,\mathrm{sen}\,f_1}{1 + e\cos f_1}; \quad \mathrm{sen}\,E_2 = \frac{\sqrt{1-e^2}\,\mathrm{sen}\,f_2}{1 + e\cos f_2} \tag{44}$$

$$M_1 = E_1 - e\,\mathrm{sen}\,E_1; \quad M_2 = E_2 - e\,\mathrm{sen}\,E_2 \tag{45}$$

$$\boldsymbol{T} = \sqrt{\frac{a^3}{m}}\left(M_2 - M_1 + 2\boldsymbol{p}N\right) \tag{46}$$

$$X_1 = \frac{S_1\cos\boldsymbol{D} - S_2}{\mathrm{sen}\,\boldsymbol{D}} + T_1; \quad X_2 = \frac{S_1 - S_2\cos\boldsymbol{D}}{\mathrm{sen}\,\boldsymbol{D}} + T_2 \tag{47}$$

$$Y = \frac{1}{\left(1-e^2\right)\mathrm{sen}\,\boldsymbol{D}}\left[3e^2\boldsymbol{T}\sqrt{\frac{m}{p^3}} - \left(\frac{2e}{q_2\,\mathrm{sen}\,f_2} - \frac{2e}{q_1\,\mathrm{sen}\,f_1}\right) + \mathrm{cotg}\,f_2 - \mathrm{cotg}\,f_1\right] \tag{48}$$

$$Z = \frac{q_2 X_2 - q_1 X_1 + \left(S_1 + S_2\right)\mathrm{tg}\,\boldsymbol{D}/2}{\mathrm{cotg}\,f_1 - \mathrm{cotg}\,f_2 + Y\left[\left(1+e^2\right)\mathrm{sen}\,\boldsymbol{D} + 2e\left(\mathrm{sen}\,f_2 - \mathrm{sen}\,f_1\right)\right]} \tag{49}$$

Solving the equation system composed by the equations 17, 20 and 21 by the Least Squares Method (cf. Rocco[7]), we obtain the transfer orbit that performs the maneuver spending a specific time but with minimum fuel consumption.

3 - RESULTS

Figures below present the results obtained. The graphs were obtained through the variation of the total time spent in the maneuver. Thus, each point was obtained executing the software to the specific time. For example, we utilized the maneuver between two terminal orbits with the following characteristics:

Initial orbit:

a_1	=	7000.00000 km;	i_1	=	0.00000 rad;	w_1	=	2.50000 rad;
e_1	=	0.10000;	p_1	=	6930.00000 km;	W_1	=	0.00000 rad.

Final orbit:

a_2	=	7100.00000 km;	i_2	=	0.00000 rad;	w_2	=	2.50000 rad;
e_2	=	0.30000;	p_2	=	6461.00000 km;	W_2	=	0.00000 rad.

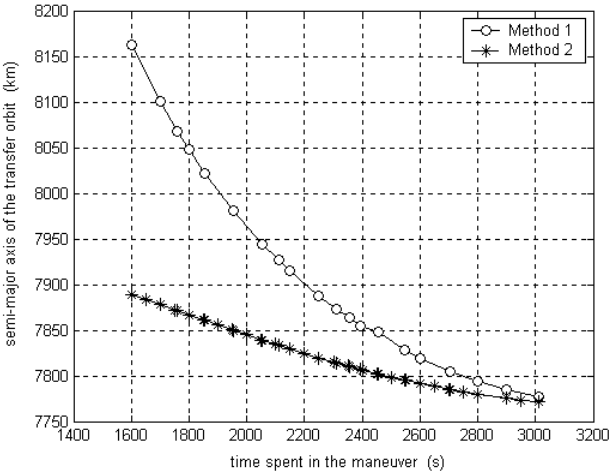


Fig. 3 – Semi-Major Axis vs. Time

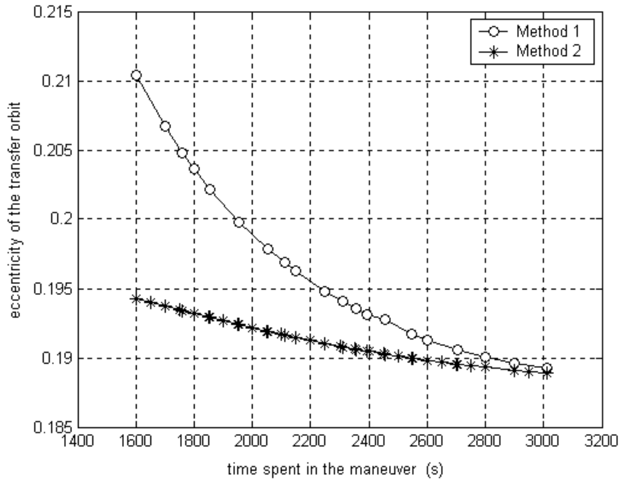


Fig. 4 – Eccentricity vs. Time

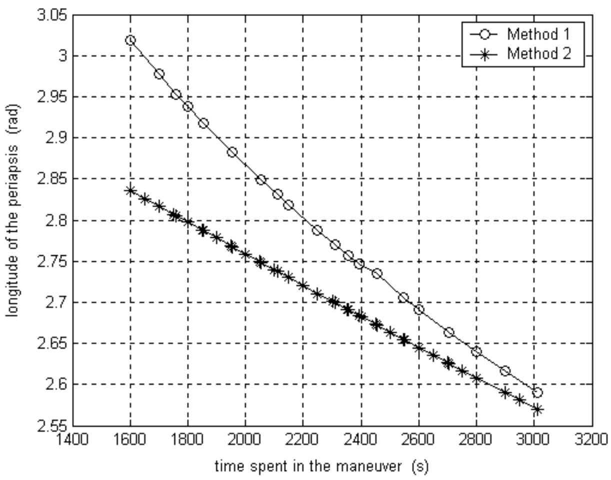


Fig. 5 – Longitude of the Periapsis vs. Time

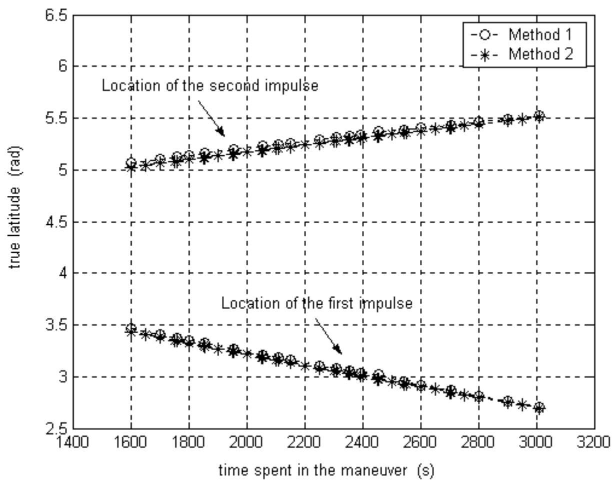


Fig. 6 – True Latitude vs. Time

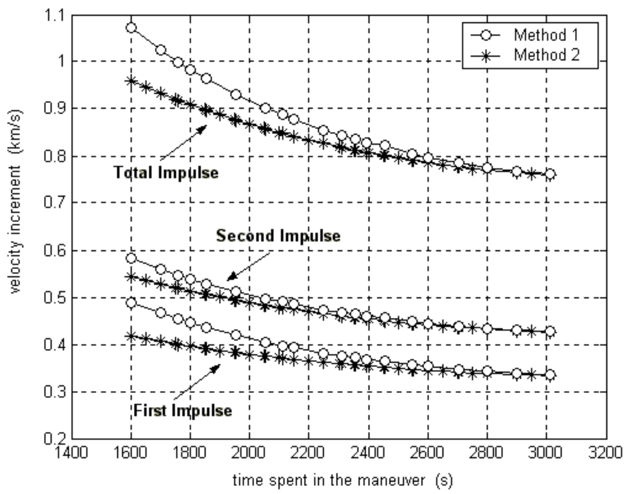


Fig. 7 – Velocity Increment vs. Time

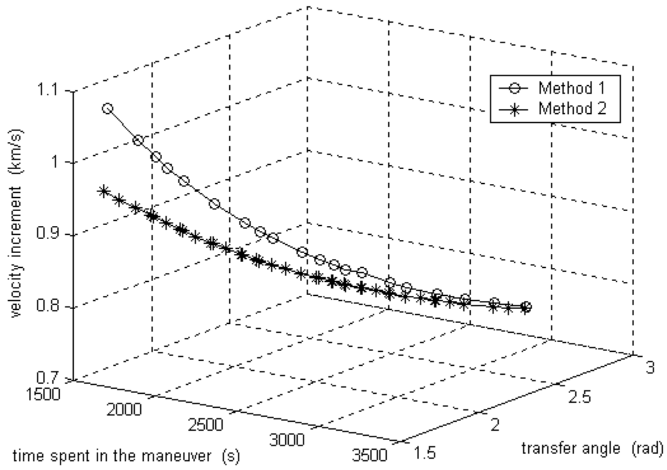
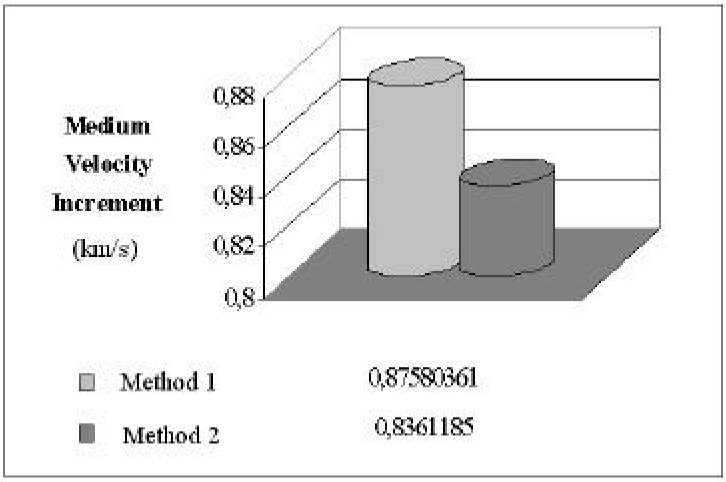


Fig. 8 – Velocity Increment vs. Transfer Angle vs. Time



$\Delta v m_{METHOD1}$ is 4,74634959 % larger than $\Delta v m_{METHOD2}$

Fig. 9 – Medium Velocity Increment.

4 - CONCLUSION

In the previous figures we can verify that the transfer orbits obtained by the application of the methods 1 and 2 are similar. In a general way, we can notice that when the maneuver spends more time the transfer angle increases while the semi-major axis, the eccentricity and the velocity increment decrease. When the maneuver spends less time the transfer angle decreases while the semi-major axis, the eccentricity and the velocity increment increase: $T \uparrow\downarrow \Rightarrow \Delta\theta \uparrow\downarrow;\ a \downarrow\uparrow;\ e \downarrow\uparrow;\ \Delta v \downarrow\uparrow$.

From the analysis of the graphs we can conclude that both methods presented satisfactory results. The solutions obtained were tested and it was verified that they are feasible. The precision demanded in both methods was of 10^{-16}, for a maximum of 1000 iterations. Thus, the solutions obtained are the roots for the systems of equations that represent the optimal conditions for the methods 1 and 2.

However, the method 2 presented better solutions than the method 1, as shown in the Figure 9. The method 1 is composed by a system of 12 non-linear equations while the method 2 is composed by a system of 3 non-linear equations. In this way, it is easier to solve the equation system of the method 2 than the equation system of the method 1. Thus, the search area of the method 2 becomes, in the practice, larger, because it is more probable that happen problems of convergence in a system with 12 equations than in a system with 3 equations. For this reason, the method 1 found a local minimum and it was not capable to find another better solutions as the method 2.

Thus, both methods were studied, implemented and tested with success. The simulations showed that the software developed can be used in real applications and it is capable to generate reliable results.

6- REFERENCES

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